On Modelling a Relativistic Hierarchical (Fractal) Cosmology by Tolman's Spacetime – I. Theory

Marcelo B. Ribeiro

Astronomy Unit

School of Mathematical Sciences

Queen Mary and Westfield College

Mile End Road

London E1 4NS

England

ABSTRACT

In this work we examine a relativistic model for the observed inhomogeneities of the Large Scale Structure where we make the hypothesis that this structure can be described as being a self–similar fractal system. The old Charlier concept of hierarchical clustering is identified with a fractal distribution and the problems raised by the use of fractal ideas in a relativistic model are discussed, as well as their relations to the Copernican and Cosmological Principles. Voids, clusters and superclusters of galaxies are assumed to be part of a smoothed–out fractal structure described by a Tolman solution. The basic concepts of the Newtonian model presented by Pietronero (1987) are reinterpreted and applied to this inhomogeneous curved spacetime. This fractal system is also assumed to have a crossover to homogeneity which leads to a "Swiss cheese" type model, composed by an interior Tolman metric and an exterior dust Friedmann solution. The Darmois junction conditions between the two spacetimes are calculated and we also obtain for the interior region the observational relations necessary to compare the model with observations. The differential equations of the interior spacetime are set up and we devise a numerical strategy for finding particular Tolman solutions representing a fractal behaviour along the past light cone.

Subject headings: cosmology: theory – galaxies: clustering – large-scale structure of the universe – relativity

1

1 Introduction

The observational analysis of the CfA redshift survey made by de Lapparent, Geller and Huchra (1986) was the first clear confirmation that the Large Scale Structure of the Universe does not show itself as a smooth and homogeneous distribution of luminous matter as was thought earlier. Rather the opposite, since up to the limits of the observations presented in their article, the 3–D cone maps show a very inhomogeneous picture, with galaxies mainly grouped in clusters or groups alongside regions devoid of galaxies, virtually empty spaces with scales of the same order of magnitude as their neighbour clusters. More recent surveys, much deeper than the previous ones, came to confirm those earlier findings presenting the Large Scale Structure as a complex mixture of interconnected voids, clusters and superclusters, observations that even led to the virtual discarding of some models which were based on the assumption that at the scale of these surveys, the Large Scale Structure would turn into a homogeneous one (Saunders et al. 1991).

If to see is to believe, the orthodox homogeneous picture seems to be in trouble when confronted with these observations, specially because a pattern appears to be common in all surveys: the deeper the probing is made, the more similar structures are observed and mapped, with clusters turning into superclusters and even bigger voids being identified.

With respect to this pattern, two ideas seem to fit in. The first is the old concept of 'hierarchical clustering' first advocated in the astronomical context by Charlier (1908, 1922) which states that galaxies join together to form clusters that form superclusters which themselves are elements of super–superclusters and so on, possibly ad infinitum. The second and more recent concept is of 'fractals', of which, for the present purpose, a rather loose tentative definition proposed by B. B. Mandelbrot seems to be adequate: "A fractal is a shape made of parts similar to the whole in some way" (see Feder 1988, p. 11).

Hierarchical cosmology has been investigated in Newtonian (Wertz 1971; Peebles 1974) and relativistic frameworks (Wesson 1978a, 1979), but the first single fractal model advanced as a description of the Large Scale Structure is due to Pietronero (1987, hereafter referred to only as Pietronero). Calzetti, Giavalisco and Ruffini (1988) followed similar arguments and investigated further implications of the fractal hypothesis for galactic clustering statistics. There have also been attempts to measure the fractal dimension of the distribution of galaxies either by assuming a single fractal approach or a multifractal one (Balian and Schaeffer 1988; Deng, Wen and Liu 1988; Jones et al. 1988; Martínez et al. 1990).

This work is an attempt to generalize Pietronero's fractal model into a relativistic framework. It differs from Ruffini, Song and Stoeger (1988) in that here we do not make use of a perturbation scheme. It is an exploratory model where some strong simplifying assumptions are made in order to avoid introducing unnecessary complications at this stage. In doing so, we shall assume that the large scale galactic clustering can be reasonably approximated by a single fractal and, hence, multifractals will not be treated here. We shall also assume relativistic dust solutions. This assumption enables us to model the smoothed–out fractal system through the general inhomogeneous dust solution due to R. C. Tolman (1934). We shall also consider a dust Friedmann spacetime as a background, as explained in section 2.

In the next section is presented a very brief summary of Pietronero's main results needed in this work, as well as the identification and discussion of basic difficulties arising when trying to apply fractal ideas in General Relativity, including their relation with the Copernican and Cosmological Principles. We also discuss how we can get around these difficulties and build up a simple model. In section 3 the observational relations of a fractal model in Tolman's spacetime are obtained and in section 4 the junction conditions between Tolman and Friedmann spacetimes are discussed. In section 5 the whole strategy and problems for solving numerically the differential equations of the model are exposed. The paper finishes with a concluding section.

2   Hierarchical clustering and fractals

As mentioned in the previous section, Pietronero presented a model for the large scale distribution of galaxies where this distribution is assumed to form a self–similar fractal structure. In this context, self–similarity means that a fractal consists of a system in which more and more structure appears at smaller and smaller scales and the structure at small scales is similar to the one at large scales (Mandelbrot 1983). It is, therefore, evident that fractals are simply a more precise version of the 'scaling' idea behind Charlier's concept of hierarchical clustering. Earlier attempts to model hierarchy started only with Charlier's hypothesis and, maybe, that is why all those models suffered a basic weakness: the lack of a precise mathematical definition for hierarchy. It is this difficulty that fractals are, it seems, able to successfully address. Basically fractals give a meaning to hierarchy.

Further to the fractal hypothesis, Pietronero defines what he calls a 'generalized mass–length relation' by starting from a point occupied by an object and counting how many objects are present within a volume characterized by a certain length scale. For a deterministic self–similar distribution, we have that within a certain radius d_0, there

are N_0 objects; then within $d_1 = kd_0$ there are $N_1 = \tilde{k}N_0$ objects; in general, within $d_n = k^n d_0$ we have $N_n = \tilde{k}^n N_0$. Generalizing this idea to a smooth relation, he then defines a relation between N and d of the type $N(d) = \sigma d^D$ where the fractal dimension $D = \log \tilde{k}/\log k$ depends only on the rescaling factors k and $\tilde{k}$ and the prefactor σ is related to the lower cutoffs N_0 and d_0, $\sigma = N_0/(d_0)^D$.

Although fractals are essentially simple, their use in a relativistic framework is not so straightforward. The difficulties start with the recognition that Pietronero's relation $N \propto d^D$ basically divides the spatial points of the system into two distinct categories: the points that belong to the fractal system where $N \propto d^D$ is valid and the ones that do not. In this sense each belonging point of the fractal system describes its remaining part by means of Pietronero's relation. In particular, any two geometrically identical portions of the fractal system carry identical number counts. A system with this property is called a 'homogeneous fractal' (Mandelbrot 1983, p. 87), though the resulting distribution over the whole space is grossly inhomogeneous. The first difficulty can be understood if we remember that the Cosmological Principle states that all observers are indistinguishable. In other words, the Cosmological Principle is realised by a continuous group of symmetry imposed upon the points of our Riemannian manifold (see e.g. MacCallum 1983). Therefore, the fractal property of dividing the space into two different categories of points runs against the realisation of a continuous group of symmetry on all points of the manifold and, consequently, a clash between fractals and the Cosmological Principle is all but unavoidable.

Such a situation, therefore, leads us to a choice between two possibilities: if one wishes to keep the Cosmological Principle one is forced to give up fractals in cosmology. On the other hand, if one is willing to accept the empirical evidence and use fractals in cosmology one must adopt a weaker interpretation of the Copernican Principle (of no preferred points in the universe) which would be compatible and applicable to fractals. In this respect, Mandelbrot (1983, p. 205) advanced the 'Conditional Cosmological Principle' which does not refer to all observers, but only to the material ones. That naturally leads to the hypothesis of a homogeneous fractal to describe galactic clustering possessing some symmetry around isolated material points which would form the fractal system. This hypothesis actually means dropping a continuous group of symmetry on all points of the manifold. By isolated points we mean points which have a neighbourhood not containing other points of the same category. Under this definition isolated points would form a subset of the Riemannian manifold. If the universe were finite, there would be a finite number of isolated points and this will in any case be true of the observable universe. In an infinite universe this number could be infinite.

We can go a step further and assume a fractal possessing spherical symme-

try around isolated points. Nevertheless, here again another difficulty comes out in that such a requirement would demand an overall property nonexistent in known geometries. Inhomogeneous spherically symmetric spaces certainly have one centre of symmetry and might also have two, which means that we would be giving up not only the Cosmological Principle but also the Copernican Principle in a cosmology with such geometries. Friedmann spacetime has spherical symmetry around infinite points and, hence, does not allow isolated points. Besides, its symmetries are such that Pietronero's relation cannot hold in its full generality, in principle $0 < D \le 3$, and, thus, we can only conclude that a 'Relativistic Fractal Cosmology' cannot be built within the Standard Friedmannian Cosmology.

The only way fractals could be seen within the context of the Standard Cosmology is if we remember that the Cosmological Principle has a statistical significance. That means the Cosmological Principle is, in practice, a statement that metric perturbations are small and this can be satisfied with density fluctuations $\delta\rho/\rho$ of some fractal types. Although this point of view may have an appeal to those who stick to Friedmannian Cosmology, it actually relegates fractals to nothing more than one possible type of local perturbations, a view already challenged by observations from the IRAS survey (Saunders et al. 1991).

Departures from the Cosmological Principle are not new. Wesson (1978b) advanced one which is somewhat related to the discussion above in the sense that he sought a formulation of the Cosmological Principle suitable to models where the density, pressure, etc, appear only in dimensionless functions solely dependent on the epoch.

In addition to these geometrical difficulties, one could argue that the observations do not contradict the possible existence of an upper cutoff of the fractal system, beyond which the distribution becomes homogeneous, though Coleman, Pietronero and Sanders (1988) claim that there is no evidence for this cutoff in the CfA survey if a different from usual statistical analysis is carried out on it.

Despite these difficulties and constraints, it is still possible to build up a simple relativistic fractal model if one adopts some sort of Einstein–Straus geometry (Einstein and Straus 1945, 1946), with the interior solution consisting of the inhomogeneous Tolman spacetime and the exterior one of the dust filled Friedmann solution. In this way, the arbitrary functions of Tolman's solution can be used to simulate a fractal system.

Modelling the Large Scale Structure in the form as described above is a different way and new combination of looking at old ideas. "Swiss cheese" type models have proved to be popular in the examination of cosmological inhomogeneities (Lake 1980;

Bonnor 1987 and references therein). Nevertheless, as far as we know the matching between Tolman and Friedmann using Darmois junction conditions was only briefly mentioned by Kantowski (1969), but without showing the calculations. The idea of using the arbitrariness in Tolman for simulation was already present in Bonnor (1972) in a more restricted model, though he did not solve the geodesic equation and, therefore, his simulation was over our present time hypersurface. As will be shown next, in this work we shall develop the model along the past light cone where the observations are actually made and using Tolman's solution in its full generality, without restrictions. Relativistic hierarchical cosmological models were attempted by Wesson (1978a, 1979), but without the fractal concept his hierarchy became ill–defined. In addition, Bonnor did not fully express his model in terms of observational quantities, relating its density to the unobservable radius coordinate (something also done by Wesson) at constant time, wherein here we adopt the opposite approach. It is, however, the analytical complexity of Tolman's solution that actually prevented its development along these lines, demanding a numerical approach as will become clear in what follows.

3   Tolman's solution as a fractal model for the distribution of galaxies

We shall approach a relativistic generalization of Pietronero's model by assuming that Tolman's solution can be used as an approximation to describe a fractal distribution of galaxies. Tolman (1934) obtained the general solutions of Einstein's equations for spherically symmetric dust in comoving coordinates which, in Bonnor's notation (Bonnor 1972), may be written (with $\Lambda = 0$ and $c = G = 1$)

$$dS^2 = dt^2 - \frac{R'^2}{f^2}dr^2 - R^2 d\Omega^2, \quad r \geq 0, \ R > 0 \tag{1}$$

where

$$d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2 \tag{2}$$

is the usual metric on the unit sphere, f is an arbitrary function of r only assumed to be of class C^2, i.e., having continuous second derivative, $R(r,t)$ satisfies

$$2R\dot{R}^2 + 2R(1 - f^2) = F \tag{3}$$

and the proper density is given by

$$8\pi\rho = \frac{F'}{2R'R^2}. \tag{4}$$

The dot means $\partial/\partial t$ and the prime means $\partial/\partial r$, and F is an arbitrary function of r also of class C^2.

The solution of equation (3) is known in the literature (Bonnor 1956, 1974) and it has three distinct cases according as $f^2 = 1$, $f^2 > 1$ and $f^2 < 1$, these cases being termed respectively parabolic, hyperbolic and elliptic models (Bonnor 1974).

In the parabolic models ($f^2 = 1$) the solution of equation (3) is

$$R = \frac{1}{2}(9F)^{1/3}(t+\beta)^{2/3}, \tag{5}$$

where $\beta(r)$ being an arbitrary function assumed of class C^2. We shall need in further calculations a second partial derivative of equation (5) and the first ones, which were obtained as follows:

$$\dot{R} = \left[\frac{F}{3(t+\beta)}\right]^{1/3}; \tag{6}$$

$$R' = \frac{1}{3}\left(\frac{9F}{t+\beta}\right)^{1/3}\left[\frac{(t+\beta)}{2F}F' + \beta'\right]; \tag{7}$$

$$\dot{R}' = \frac{1}{9}\left(\frac{9F}{t+\beta}\right)^{1/3}\left(\frac{F'}{F} - \frac{\beta'}{(t+\beta)}\right). \tag{8}$$

In the hyperbolic models ($f^2 > 1$) the solution of equation (3) may be written in terms of a parameter Θ,

$$R = \frac{F(\cosh 2\Theta - 1)}{4(f^2 - 1)}, \tag{9}$$

$$t + \beta = \frac{F(\sinh 2\Theta - 2\Theta)}{4(f^2 - 1)^{3/2}} \tag{10}$$

and these quantities' derivatives can be found as

$$\dot{R} = \left(\frac{\sinh 2\Theta}{\cosh 2\Theta - 1}\right)\sqrt{f^2 - 1}, \tag{11}$$

$$R' = \left[\frac{1}{4(A-1)(f^2-1)^2}\right]\left[(4A + B^2 - 6B\Theta - 4)Fff' - \right.$$

$$\left. -2F'(A - B\Theta - 1)(f^2 - 1) + 4B\beta'(f^2 - 1)^{3/2}\right], \tag{12}$$

7

$$\dot{R}' = \frac{1}{F(f^2-1)\left[(3B^2+4)-A(B^2+4)\right]}\left\{\sqrt{f^2-1}\left[(5B-6\Theta)A-\right.\right.$$

$$\left.-B^3-5B+6\Theta\right]Fff'-\left[F'\sqrt{f^2-1}(B-2\Theta)-\right.$$

$$\left.\left.-4\beta'(f^2-1)^2\right](A-1)(f^2-1)\right\}, \tag{13}$$

where

$$A \equiv \cosh 2\Theta, \qquad B \equiv \sinh 2\Theta. \tag{14}$$

Finally, in the elliptic models ($f^2 < 1$) a parameter Θ is again needed to write the solution of equation (3)

$$R = \frac{F(1-\cos 2\Theta)}{4|\,f^2-1\,|}, \tag{15}$$

$$t + \beta = \frac{F(2\Theta - \sin 2\Theta)}{4|\,f^2-1\,|^{3/2}}, \tag{16}$$

whose derivatives are

$$\dot{R} = \left(\frac{\sin 2\Theta}{1-\cos 2\Theta}\right)\sqrt{|\,f^2-1\,|}, \tag{17}$$

$$R' = \left[\frac{1}{4(A-1)|\,f^2-1\,|^2}\right]\left[(4A-B^2+6B\Theta-4)Fff'+\right.$$

$$\left.+2F'(A+B\Theta-1)\,|\,f^2-1\,|-4B\beta'|\,f^2-1\,|^{3/2}\right], \tag{18}$$

$$\dot{R}' = \frac{1}{F\,|\,f^2-1\,|\,[A(B^2-4)+4-3B^2]}\left\{\sqrt{|\,f^2-1\,|}\left[(5B-6\Theta)A+\right.\right.$$

$$\left.+B^3-5B+6\Theta\right]Fff'+\left[F'\sqrt{|\,f^2-1\,|}(B-2\Theta)+\right.$$

$$\left.\left.+4\beta'|\,f^2-1\,|^2\right](A-1)\,|\,f^2-1\,|\right\}, \tag{19}$$

where

$$A \equiv \cos 2\Theta, \qquad B \equiv \sin 2\Theta. \tag{20}$$

In order to make use of Tolman's models as descriptors of observations, it is necessary first of all to adopt the appropriate definition of distance of a radiating

source, which in this case will be assumed to be the 'luminosity distance' as that is the definition generally used by observers in their data analysis. Its expression can be obtained by calculating first the 'observer area distance' r_0 (see Ellis 1971; this is the same as the 'corrected luminosity distance' of Kristian and Sachs 1966 and also the same as the 'angular diameter distance' of Weinberg 1972)

$$(r_0)^2 = \frac{dS_0}{d\Omega_0} = \frac{R^2 d\theta \sin\theta d\phi}{d\theta \sin d\phi} = R^2 \tag{21}$$

in the spacetime (1). Here $d\Omega_0$ is the solid angle subtended by a bundle of null geodesics diverging from the observer and dS_0 the cross–sectional area of this bundle at some point. Further, it was shown by Ellis (1971) that the luminosity distance d_l and the observer area distance are related by

$$(d_l)^2 = (r_0)^2 (1+z)^4 \tag{22}$$

which implies

$$d_l = R(1+z)^2 \tag{23}$$

in Tolman's spacetime. Here z is the 'redshift' of a source as measured by the observer.

The next step in applying Pietronero's procedure in Tolman's spacetime is to obtain the expression for 'number counts'. In any cosmological model if we consider a small affine parameter displacement $d\lambda$ at some point P on a bundle of past null geodesics subtending a solid angle $d\Omega_0$, and if n is the number density of radiating sources per unit proper volume, then the number of sources in this section of the bundle is (Ellis 1971)

$$dN = (r_0)^2 d\Omega_0 \left[n(-k^a u_a) \right]_P d\lambda \tag{24}$$

where k^a is the propagation vector of the radiation flux and u^a is the 4–velocity of the observer. Assuming a comoving observer $u^a = (1,0,0,0)$ and that the past null geodesic is a radial one, given by

$$\frac{dt}{d\lambda} = -\left(\frac{R'}{f}\right)\frac{dr}{d\lambda}, \tag{25}$$

and also remembering spherical symmetry, equation (24) becomes

$$dN = 4\pi n \frac{R' R^2}{f} dr. \tag{26}$$

We shall also assume that the sources are mostly galaxies, with rest masses of $M_G \sim 10^{11} M_\odot$ and, therefore, equation (4) allows us to write

$$n = \frac{\rho}{M_G} = \frac{F'}{16\pi M_G R' R^2}. \tag{27}$$

9

Once we substitute equation (27) into equation (26) and integrate the latter, we obtain the number $N_c(r)$ of sources which lie at radial coordinate distances less than r as seen by the observer at $r = 0$

$$N_c(r) = \frac{1}{4M_G} \int_C \frac{F'}{f} dr, \qquad (28)$$

where the integration is made along the curve C formed by the past light cone parametrized by r. Two notes should be made about the equation above. Firstly, the affine parameter λ becomes implicit, a fact which brings advantages in carrying out numerical calculations. Secondly, if we let $t(r)$ be the solution of the geodesic C, which is given by equation (25), we can see that although equation (28) does not have the time coordinate explicitly in the right hand side, the integration is along the geodesic where $R = R[r, t(r)]$. That is because equation (25) was used in the derivation of equation (28).

Now in order to make the appropriate definition of density applicable to a fractal model, we will follow Wertz (1971) and Bonnor (1972) and distinguish between a 'volume density' ρ_v obtained by averaging over a sphere of given volume and the 'local density' ρ given by equation (4). Nevertheless, our definition of volume density is different from the latter inasmuch as in this model we use the luminosity distance as our definition of distance, a fact that basically means that we observe distances in a curved spacetime as if this spacetime were a Euclidean one. In other words, d_l is the distance which the source would be at if it were stationary in a Euclidean space. In this sense, therefore, the volume of the sphere which contains the sources may be written as

$$V(r) = \frac{4}{3}\pi(d_l)^3 = \frac{4}{3}\pi R^3(1+z)^6 \qquad (29)$$

and the volume density is given by

$$\rho_v(r) = \frac{M_G N_c(r)}{V(r)} = \frac{3}{16\pi R^3(1+z)^6} \int_C \frac{F'}{f} dr. \qquad (30)$$

This expression merely states the volume density in Tolman's spacetime and does not contain by itself any relationship to a fractal distribution of dust. Therefore, following Pietronero's hypothesis for a self–similar fractal distribution[1], if within a certain radius $(d_l)_0$ there are $(N_c)_0$ objects and then within $(d_l)_1$ there are $(N_c)_1$ objects, we can then write a smoothed–out relation between N_c and d_l as

$$N_c = \sigma(d_l)^D, \qquad (31)$$

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[1]Here the self–similarity due to fractals should not be confused with the one due to homothetic Killing vectors. The latter is discussed in Cahill and Taub (1971).

where σ is a constant related to the lower cutoffs $(N_c)_0$ and $(d_l)_0$ of the distribution and D is its fractal dimension that can be noninteger. This is the natural generalization of Pietronero's definition originally made in a Newtonian context.

We must point out that the adoption of equation (31) is the obvious thing to do if one wishes to follow the astronomical procedure and compare the model with observations. Nevertheless, fractal dimensions have so far been defined in Euclidean spaces and it is not at all clear whether equation (31) is the most appropriate definition to take in curved spacetimes. We can see a possible shortcoming if we remember that it is usually assumed that in a homogeneous distribution $D \cong 3$ (Mandelbrot 1983, Pietronero) and one could argue that this would be the value to be found for Friedmann. However, Friedmann spacetime is homogeneous at constant time coordinates and when we integrate along the past light cone, going through hypersurfaces of t constant with each one having different values for the density, it should not be so surprising if D departs from the value 3 even in a spatially homogeneous spacetime.

From equation (31) and also considering equations (29) and (30), it is possible now to compute the volume density for a sphere of certain radius that contains a portion of the fractal distribution:

$$\rho_v = \frac{3\sigma M_G}{4\pi}(d_l)^{-\gamma}, \quad \gamma = 3 - D. \tag{32}$$

This is the same sort of expression as obtained by de Vaucouleurs (1970) when he argued in favour of a hierarchical cosmology. If we now take the volume density (30) and substitute into equation (32) we get

$$\int_C \frac{F'}{f}dr = 4\sigma M_G \big[R(1+z)^2\big]^D. \tag{33}$$

This is the condition that the three arbitrary functions $f(r)$, $F(r)$ and $\beta(r)$ must satisfy such that a fractal distribution of galaxies is simulated in Tolman's spacetime. We can call equation (33) the 'self–similar condition' as it is clearly the particular case of the general equation (31) when applied to Tolman's solution.

As the final issue before the end of this section, although the redshift is essential in all previous expressions, it has not been explicitly calculated for the spacetime under consideration. In order to do so, let us start with the general expression for the redshift (see e.g. Ellis 1971)

$$1 + z = \frac{(u^a k_a)_{\text{source}}}{(u^b k_b)_{\text{observer}}}. \tag{34}$$

We shall assume that both source and observer are comoving and, hence, equation (34) becomes

$$1 + z = \frac{dt}{d\lambda}\bigg|_{\lambda=\lambda_0} \left(\frac{dt}{d\lambda}\bigg|_{\lambda=0}\right)^{-1} \tag{35}$$

11

where λ_0 is any value taken by the affine parameter λ along the geodesic. We shall make use of the condition that the spacetime should be regular at the spatial origin and, therefore, when $r \to 0$, $R = r$, $f = 1$, $R' = 1$, $F = 0$ (Bonnor 1974). These conditions together with equation (25) allow us to write

$$1 + z = \left.\frac{R'}{f}\frac{dr}{d\lambda}\right|_{\lambda=\lambda_0} \left(\left.\frac{dr}{d\lambda}\right|_{\lambda=0}\right)^{-1}. \tag{36}$$

Following an idea suggested by M. A. H. MacCallum, the right hand side of equation (36) can be calculated by starting with the Lagrangian of the radial metric

$$L = \left(\frac{dS}{d\lambda}\right)^2 = \left(\frac{dt}{d\lambda}\right)^2 - \left(\frac{R'}{f}\frac{dr}{d\lambda}\right)^2. \tag{37}$$

The Lagrange equations of second kind

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\nu} - \frac{\partial L}{\partial x^\nu} = 0, \quad \dot{x}^\nu = \frac{dx^\nu}{d\lambda}$$

applied to equation (37) result in

$$\frac{d^2t}{d\lambda^2} + \left(\frac{dr}{d\lambda}\right)^2 \frac{R'\dot{R}'}{f^2} = 0, \tag{38}$$

$$\frac{d^2r}{d\lambda^2} + \frac{1}{R'}\left(\frac{dr}{d\lambda}\right)^2\left(R'' - \frac{f'R'}{f}\right) + 2\frac{dr}{d\lambda}\frac{dt}{d\lambda}\frac{\dot{R}'}{R'} = 0. \tag{39}$$

Here in the second equation the assumption that $(R')^2 \neq 0$ was made (otherwise $g_{rr} = 0$).[2] Considering the radial null geodesic it is possible to integrate equations (38) and (39) once, obtaining

$$\frac{dt}{d\lambda} = \frac{1}{I + C_1}, \tag{40}$$

$$\frac{dr}{d\lambda} = \left[\int\left(\frac{R''}{R'} - \frac{f'}{f} - \frac{2\dot{R}'}{f}\right)d\lambda + C_2\right]^{-1} \tag{41}$$

where

$$I \equiv \int \frac{\dot{R}'}{R'}d\lambda \tag{42}$$

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[2]Actually the boundary surfaces on which $R' = 0$ are shell crossings, where the density ρ diverges and the region beyond has negative density. They indicate a breakdown of the basic assumptions of the Tolman metric (see Hellaby and Lake 1985, 1986 for details).

and C_1, C_2 are two integration constants whose relationship can be found by substituting equations (40) and (41) back into the geodesic:

$$\int \left(\frac{R''}{R'} - \frac{f'}{f} - \frac{2\dot{R}'}{f} \right) d\lambda + C_2 = -\frac{R'}{f}(I + C_1). \tag{43}$$

This equation is valid for any λ, including the point $\lambda = 0$ (λ is taken to be zero at $r = 0$) where the regular conditions make equation (43) become

$$C_2 = -C_1. \tag{44}$$

The same conditions substituted into equation (40) lead to

$$\left. \frac{dt}{d\lambda} \right|_{\lambda=0} = \frac{1}{C_1}. \tag{45}$$

However, as our observations are along the past null geodesic, the natural choice for C_1 is

$$C_1 = -1 \quad \Longrightarrow \quad C_2 = 1, \tag{46}$$

which considering equation (43) implies that equations (40) and (41) may be written as

$$\frac{dt}{d\lambda} = \frac{1}{I-1}, \tag{47}$$

$$\frac{dr}{d\lambda} = \frac{f}{(1-I)R'}. \tag{48}$$

The redshift can, therefore, be calculated once we again make use of the regularity conditions on equation (48) to get

$$\left. \frac{dr}{d\lambda} \right|_{\lambda=0} = 1 \tag{49}$$

that substituted into equation (36), together with equation (48), gives

$$z = \frac{I}{1-I}. \tag{50}$$

The integral I still explicitly contains the affine parameter, which can be made implicit by considering equation (48) and differentiating equation (42)

$$\frac{dI}{dr} = \frac{\dot{R}'}{f}(1-I). \tag{51}$$

Hence, the redshift in equation (50) needs the solution of the differential equation above for the values of I.

As a final remark, it is of great numerical advantage that equation (51) is written only in terms of the radial coordinate r and, therefore, can be solved simultaneously with the past radial null geodesic (25). In this form the redshift becomes an implicit function of r only, $z = z[I(r)]$.

4   The matching to a dust Friedmann exterior

As discussed in section 2, the fractal system is assumed to have a crossover to homogeneity, which will be represented in this model by the matching between the inhomogeneous Tolman metric and the homogeneous Friedmann one. In order to achieve a smooth transition it is necessary to solve the junction conditions for the two metrics. In this case this is a straightforward calculation in view of the fact that both metrics are comoving dust filled spherically symmetric spacetimes.

Let us start by writing the Friedmann metric as

$$d\overline{S}^{\,2} = dT^2 - a^2(T)\left[dx^2 + g^2(x)d\overline{\Omega}^{\,2}\right] \tag{52}$$

where

$$d\overline{\Omega}^{\,2} \equiv d\overline{\theta}^{\,2} + \sin^2\overline{\theta}\,d\overline{\phi}^{\,2},$$

$$g(x) = \begin{cases} \sin x, & K = +1, \\ x, & K = 0, \\ \sinh x, & K = -1, \end{cases}$$

and $a(T)$ satisfies the Friedmann equation

$$\dot{a}^2 = \frac{8\pi}{3}\mu a^2 - K. \tag{53}$$

Here the dot means $\partial/\partial T$ and the prime $\partial/\partial x$, and μ is the dust density.

Let Σ be a hypersurface which separates Riemannian spacetime into two four–dimensional manifolds V^- and V^+ (Israel 1966, 1967). Here V^- is the interior Tolman metric and V^+ the exterior Friedmann one. The hypersurface Σ is then defined by

$$\Sigma_- = r - \Sigma_0 = 0, \qquad \Sigma_+ = x - \Sigma_0 = 0 \tag{54}$$

where the indexes $+$ and $-$ mean the approach to Σ is from V^- or V^+, and Σ_0 is the constant which defines the end of the Tolman cavity.

The Darmois junction conditions state that V^- and V^+ match across Σ if the first and second fundamental forms of Σ are identical (Bonnor and Vickers 1981). As V^+ and V^- are spherically symmetric, it is natural to take the intrinsic metric to Σ, $dS_\Sigma^2 \equiv g_{\alpha\beta}d\xi^\alpha d\xi^\beta$, $(\alpha = 0, 2, 3)$, also to be spherically symmetric, where ξ^α are the intrinsic coordinates of Σ. In this case $\xi^\alpha = x_+^\alpha = x_-^\alpha$ and, hence, $\xi^0 = t = T$, $\xi^2 = \theta = \overline{\theta}$, $\xi^3 = \phi = \overline{\phi}$. Thus the first fundamental form identity of Σ, $dS_-^2 = dS_+^2$, leads to

$$R = ag \qquad \text{on} \quad \Sigma. \tag{55}$$

The unit normals to Σ, $n_a{}^{\pm} = \left[\Sigma_{,a}\left(-g^{bc}\,\Sigma_{,b}\Sigma_{,c}\right)^{-1/2}\right]^{\pm}$, directed from V^- to V^+, $(a = 0,1,2,3)$, are needed to calculate the condition for continuity of the second fundamental form or extrinsic curvature. For V^- and V^+ they are respectively

$$n_b{}^- = \frac{R'}{f}\,\delta_b^1, \qquad n_c{}^+ = a\,\delta_c^1. \tag{56}$$

The extrinsic curvature $K_{ab} = n_{a;b}$ on Σ takes the form

$$K_{\alpha\beta} = \frac{\partial x^a}{\partial \xi^\alpha}\frac{\partial x^b}{\partial \xi^\beta}K_{ab} \tag{57}$$

and the explicit calculation of the condition $K_{\alpha\beta}{}^- = K_{\alpha\beta}{}^+$ gives

$$Rf = agg' \qquad \text{on} \quad \Sigma. \tag{58}$$

Substituting the first condition into the second leads to

$$f = g' \qquad \text{on} \quad \Sigma. \tag{59}$$

We can check whether these results are correct if we remember that both spacetimes are equivalent (spherically symmetric and dust filled) and, therefore, Tolman metric should change to the Friedmann one on Σ. This is easily verified if we substitute equation (55), its radial coordinate derivative and equation (59) into equation (1).

The junction conditions above have also an effect on the gravitational mass within the Tolman cavity. If we interpret equation (3) as a total energy equation (Bondi 1947), we may define the gravitational mass inside r as

$$m(r) \equiv \frac{F(r)}{4} = \int_0^r 4\pi\rho R' R^2 dr \tag{60}$$

which implies that equation (3) may be written as

$$\frac{\dot{R}^2}{2} - \frac{m}{R} = \frac{1}{2}(f^2 - 1). \tag{61}$$

Therefore, the total gravitational mass trapped by Σ within the Tolman cavity is

$$M = m(\Sigma_0) = \int_0^{\Sigma_0} 4\pi\rho R' R^2 dr. \tag{62}$$

If we apply the junction conditions to the equation above we get

$$M = \int_0^{\Sigma_0} 4\pi\rho R' R^2 dr = \int_0^{\Sigma_0} 4\pi\mu a^3 g^2 g' dx = \frac{4\pi}{3}\mu a^3 g^3(\Sigma_0) = \overline{M} \tag{63}$$

where $\overline{M}$ is the gravitational mass of the Friedmann metric for the region $0 < x \leq \Sigma_0$. Therefore, the matching restricts the mass inside V^-. The gravitational mass must be the same as if the whole spacetime were Friedmannian and the Tolman cavity were never there. If there is any overdensity within the cavity, there must also be underdensity before the uniform region is reached, in order that the average densities will be the same.

This constraint appears to imply that Einstein–Straus like geometries are too restricted to be used for understanding the nature of the real inhomogeneous universe, as the inhomogeneities will be severely restricted in a way that could be taken to be unnecessary. However, it has already been pointed out by Ellis and Jaklitsch (1989) that the matching can be used as a 'fitting condition' specifying what is an appropriate Friedmann model to use as a background in a given lumpy universe model. In other words, if we can measure the mass distribution in our neighbourhood, that tells us whether the Friedmann background has enough mass to be a closed or open model. Hence, the matching conditions are interpreted not as a handicap but as advantageous cosmological fitting conditions, ensuring that the Friedmann model overall mass is correctly adapted to the inhomogeneous universe.

5   Numerical methods

In section 3 the necessary expressions for modelling a fractal dust in Tolman's spacetime were developed and it was shown that excluding the number counting, all other relevant observational relations can be computed if we know the solutions of the two linear first order ordinary differential equations: the radial past null geodesic and the equation for the redshift

$$\frac{dt}{dr} = -\frac{R'}{f}, \qquad \frac{dI}{dr} = (1 - I)\frac{\dot{R}'}{f}. \tag{64}$$

A brief inspection of the expressions for R' and $\dot{R}'$ shows that an attempt to find an analytical solution for these equations is virtually hopeless, specially in the elliptic and hyperbolic models and, hence, a numerical approach is made necessary. Let us suppose that $t(r)$ is the solution of the geodesic and $I(r)$ of the equation for the redshift. The observations lie along the past light cone and in order to compare the numerical results with them, R' and $\dot{R}'$ must be evaluated along the geodesic. Therefore, we must compute $R = R[r, t(r)]$ and its derivatives, which means that $I(r)$ can only be found if $t(r)$ is already known.

We shall assume that 'here and now' is defined by $r = 0$, $t = 0$, $\lambda = 0$, definition

which implies the initial conditions for (64) as being

$$t(0) = 0, \qquad I(0) = 0. \tag{65}$$

This assumption, however, runs into trouble because due to the regular condition $F(0) = 0$, at the origin of the elliptic and hyperbolic models the parameter Θ remains undefined. This difficulty can be overcome if we make the hypothesis that the metric remains flat from $r = 0$ till some small value $r = \varepsilon$, and beyond it the spacetime changes to a curved one. Hence, we replace the initial conditions (65) by

$$t(\varepsilon) = -\varepsilon, \qquad I(\varepsilon) = 0. \tag{66}$$

In the previous sections it was explicitly assumed that the fractal dust under consideration has a lower cutoff associated with the constant σ of equation (31), below which this structure is no longer observed. At the Galactic level no fractal distribution is observed and, therefore, we can naturally assume that this structure starts at least at the scale of the Local Group, which would mean taking $\varepsilon \stackrel{<}{\sim} 1$ Mpc.

The goal of modelling Tolman's solution to a fractal distribution is to make use of the freedom of the arbitrary functions in order to find out particular functions $f(r)$, $F(r)$, $\beta(r)$ such that the volume density takes the de Vaucouleurs' density power law (32). The self–similar condition (33) is of little practical use because its right hand side cannot be computed analytically. In these circumstances, the following numerical strategy was devised: we carry out the discretisation of the radial coordinate r_i $(i = 1, 2, \ldots, n; \ \varepsilon \leq r \leq \Sigma_0)$ and for each set of points r_i, t_i, I_i and r_{i+1} we calculate t_{i+1} then I_{i+1} through some numerical algorithm for solving ordinary differential equations. Also knowing r_i, equation (28) permits the computation of N_{c_i} by means of a numerical quadrature. In the elliptic and hyperbolic cases it is also necessary to use a root–finding algorithm to evaluate Θ_i. With these results it is possible to compute the observational quantities d_{l_i}, ρ_{v_i} and z_i through equations (23), (30) and (50) respectively. These values immediately allow us to plot graphics relevant to observations like number counting versus redshift.

As these calculations will produce a great quantity of numbers, it is necessary here a direct method of checking whether a fractal distribution was modelled, specially because we might not easily see a true power law like expression for the volume density against the luminosity distance. For this purpose, we can simply take the logarithm of equation (32)

$$\log \rho_v = a_1 + a_2 \log d_l \tag{67}$$

and carry out a linear fitting over the points obtained through numerical integration. Naturally, at each integration a particular set of functions $f(r)$, $F(r)$, $\beta(r)$, is chosen

beforehand and if the fitting is successful (measured by an acceptable goodness of fit), that is, if the results have a linear form given by equation (67) with negative slope, a fractal distribution of dust was modelled by a particular Tolman's spacetime. If the fitting is not successful a new attempt is made with a different set of functions. That method is tantamount to a numerical simulation procedure for modelling a fractal distribution of dust by Tolman's spacetime.

Once the fitting is successful, the fractal dimension D and the constant σ can be found directly from the fitted constants in equation (67) as

$$D = a_2 + 3 \tag{68}$$

and

$$\sigma = \frac{4\pi}{3M_G} \exp(a_1). \tag{69}$$

As stated above, in the hyperbolic and elliptic models for each $t_i(r_i)$, $\beta_i(r_i)$, $F_i(r_i)$, $f_i(r_i)$ we need to find the root Θ_i of equations (10) or (16) in order to be able to evaluate R_i, $\dot{R}_i$, R'_i and $\dot{R}'_i$. That is done numerically by finding an interval where the root lies, then using some algorithm to hunt it down. That interval obviously must be limited to the physical regions of the spacetime under consideration and, therefore, the following remark must be made. The function $\beta(r)$ determines the local time at which $R = 0$ and, consequently, the hypersurface $t + \beta = 0$ is a surface of singularity. In view of this the physical region to be considered is defined by $t + \beta > 0$.

Bearing this point in mind, we can now proceed with the bracketing of the roots. In the elliptic case due to the boundness of the sine function it is easy to see that

$$\frac{4}{F}(t+\beta)|\, f^2 - 1\,|^{3/2} - 1 \leq 2\Theta \leq \frac{4}{F}(t+\beta)|\, f^2 - 1\,|^{3/2} + 1. \tag{70}$$

The hyperbolic case is a bit more complicated as the hyperbolic sine is not bounded. Let us write equation (10) as

$$G(\Theta) = \sinh 2\Theta - 2\Theta - \frac{4}{F}(t+\beta)(f^2 - 1)^{3/2} = 0. \tag{71}$$

The function $G(\Theta)$ changes sign within the interval $[G(0), G(+\infty)]$ which is where the root lies ($F \geq 0$ otherwise we would have negative gravitational mass). As $G(0) < 0$ and $G(+\infty) > 0$, the change of sign occurs when the inequality $G(\Theta) > 0$ is satisfied for $\Theta > 0$. Using a power series expansion for $\sinh 2\Theta$, this inequality can be written as

$$\frac{(2\Theta)^3}{3!} + \frac{(2\Theta)^5}{5!} + \frac{(2\Theta)^7}{7!} + \ldots > \frac{4}{F}(t+\beta)(f^2 - 1)^{3/2}. \tag{72}$$

If $\Theta \geq 1$ the inequality will be satisfied provided the smallest term of the series is bigger than the right hand side of equation (72). If $0 < \Theta < 1$ the first term of the series

18

will dominate and the inequality will be satisfied provided this first term is bigger than the right hand side of equation (72). In short, for the hyperbolic models the root of equation (10) lies within the interval

$$0 < \Theta \leq \left[\frac{3}{F}(t + \beta)(f^2 - 1)^{3/2} \right]^{1/3}. \tag{73}$$

6 Conclusion

In this work we have proposed a relativistic hierarchical (fractal) cosmology in the inhomogeneous Tolman spacetime based on the reinterpretation and relativistic generalization of Pietronero's Newtonian model. We have assumed that the large scale distribution of galaxies forms a homogeneous fractal system and discussed how fractals give a new and precise meaning to Charlier's concept of hierarchical clustering. We concluded that the fractal property of dividing the space in points of different categories clashes with the Cosmological Principle, a fact which led us to seek a weaker interpretation of the Copernican Principle. In doing so we have assumed Mandelbrot's Conditional Cosmological Principle and made the hypothesis of a fractal with the property of being spherically symmetric around isolated points. Such a fractal, however, demands an overall property nonexistent in known geometries.

Considering these difficulties, we have advanced a simple exploratory model compatible with the Conditional Cosmological Principle by adopting a version of Einstein–Straus geometry consisting of an interior inhomogeneous Tolman spacetime and an exterior Friedmann one. Our fractal system is smoothed–out and has an upper cutoff which coincides with the end of the Tolman cavity. We have obtained the observational relations for the Tolman spacetime necessary to compare the model with the astronomical observations, namely the luminosity distance, number counts, volume density (average density) and redshift. We have also found a self–similar condition which the arbitrary functions of Tolman spacetime must satisfy in order to simulate a fractal dust. The Darmois junction conditions between the two spacetimes were also calculated.

The differential equations necessary for evaluating the observational relations were set up and we have discussed a numerical approach for solving these equations, inasmuch as that is virtually impossible to be done analytically. In this respect the numerical method consists of choosing particular Tolman's solutions and carrying out a linear fitting over the points obtained through numerical integration. That aims to see whether or not these particular solutions obey a de Vaucouleurs like power law relation for the volume density and then to find the fractal dimension of the distribution.

Finally, in the physical region of the Tolman spacetime we found the interval where the parameter Θ lies in the elliptic and hyperbolic cases.

The numerical results of this model are the subject of a forthcoming paper.

Acknowledgments

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